Costly Enforcement in Credit Economies^{*}

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Abstract

In a monetary model based on Lagos and Wright (2005) where unsecured credit and money are used as means-of-payments, we analyse how the cost and quality of the record-keeping technology affect welfare. Specifically, monitoring agents' debt repayment is costly but is essential to the use of unsecured credit because of limited commitment. To finance this cost, fees on credit transactions are imposed, and the maximum credit limit that is incentive compatible depends on such fees and monitoring level. Alternatively, the use of money avoids such costs. A higher credit limit does not necessarily improve welfare, especially when the limit is high: the benefit from increased trade surpluses from a higher credit limit is offset by increased cost of monitoring to achieve the increase. Moreover, under the optimal arrangement, optimal credit limit decreases with the marginal cost of monitoring, and, when the marginal cost is high, it is optimal to have a pure-currency economy. However, there can be a non-monotonic relationship between the optimal record-keeping level and the optimal credit limit.

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1 Introduction

Though technological progress in recent decades has greatly enhanced record keeping, a crucial ingredient to sustain credit arrangements (Kocherlakota and Wallace, 1998), it does not eliminate the basic friction that hinders credit transactions, namely, limited commitment of economic agents. Moreover, the cost to establish a modern credit system which allows for accurate updates of transaction data is still non-trivial. In contrast, an economy can avoid this costly credit arrangement by using money as the only payment instrument. Indeed, in many economies today, most transactions are accomplished with cash alone.¹ These observations naturally lead to two research questions when we consider endogenous recording keeping and enforcement: first, when is it feasible to adopt the technology and sustain a credit economy? second, when the use of credit is sustainable, would it be socially optimal to use it?

In this paper, we propose a monetary model based on Lagos and Wright (2005) with endogenous liquidity needs and introduce a costly and incomplete record-keeping technology to answer these questions. Specifically, we endogenize the level of record-keeping and study the socially optimal use of payment instruments when agents cannot commit to repay their future obligations. Previous literature, as Bethune et al. (2018), has shown that in an environment with complete record-keeping and limited commitment, a pure credit economy is better than a pure currency economy in terms of welfare, a result consistent with Kocherlakota (1998) in more general settings.² The gain in welfare is as agents bear opportunity cost of holding fiat money and thereby bring an inefficient level of liquidity to trade, credit arrangement relaxes this liquidity constraint and hence improves gains from trade. Thus, when the cost of record-keeping is zero, it is optimal to use credit. But once we introduce the cost of recording keeping, where better enforcement requires more real resources from the

¹See The 2020 McKinsey Global Payment Report for the cash usages by country, where in emerging economies, such as Argentina, India, Indonesia, Malaysia and Mexico, cash accounts for an average 80 percent of total transaction volume. Also see the Fifth Global Payment System Survey by World Bank for the cashless transaction per capita according to country income levels.

²Bethune et al. (2018) show this by assuming a fixed bargaining protocol across the pure currency and the pure credit economies.

society, this comparison would then depend on the marginal cost of providing better quality record-keeping service.

Indeed, by endogenizing the record-keeping technology, we capture a non-trivial trade-off that is missing in the literature: while a record-keeping of higher quality directly increases liquidity provided by the credit system, its cost requires real resource which would affect liquidity provision indirectly. Unsurprisingly, credit equilibrium is sustainable only when the marginal cost to improve record-keeping is not too high. However, we find that for a range of intermediate marginal costs, there can be a non-monotonic relationship between credit limits and welfare. Moreover, for a range of such marginal costs, a pure monetary economy generates higher welfare than a credit economy, even though a higher level of liquidity can be sustained in a credit economy.

Following Hu et al. (2009) and Andolfatto (2013), we assume that the government has limited coercion power, and the cost of record-keeping has to be borne by private agents in a way that this burden also affects liquidity provision due to limited commitment. Our model shares the standard environment according to Lagos-Wright, where agents meet bilaterally in a decentralized market and use money and/or credit to trade. This decentralized market is followed by a centralized market where agents repay obligations and rebalance money holding. Following Kocherlakota and Wallace (1998) and Bethune et al. (2015), we introduce a record-keeping system which is subject to stochastic record-updating, that is, it may fail to record a default when it happens and it is costly to lower the rate of such failures.

Coupled with the limited commitment friction, the accuracy of the record-keeping technology implies a level of self-enforcement that is linked to the incentive compatible credit limit. To finance a more accurate record-keeping system and hence a higher level of enforcement, agents who access the technology to issue credit have to pay a fee that is used to finance the cost of operating the record-keeping technology. Thus, in principle, a higher level of enforcement does not necessarily imply a higher debt limit, as the cost of better record keeping technology also enters the incentive constraints for repayments. For low marginal costs of record-keeping, however, we show that it is always monotonic, but this may not be true for higher marginal costs. Our main result characterizes the endogenous record-keeping technology for a given marginal cost, and how the optimal means-of-payments and liquidity provision varies with that cost. For a given marginal cost of improving record-keeping, we find that the welfare can be non-monotonic in the credit limit. In particular, for low credit limit below the amount of liquidity that money alone can provide, higher credit limit would only require higher social cost for better enforcement without increasing the overall liquidity, and hence would lower welfare. This result is a natural consequence of the result in Gu et al. (2016) when we add cost to the record keeping system. However, while welfare would increase with credit limit after that, it might decrease again when the credit limit is high. The intuition is that the marginal benefit from an increase in credit limit becomes smaller as the limit increases, but the marginal cost that is required remains constant or becomes higher. As a result, the optimal credit limit may not be the highest implementable one.

Our second main result then characterizes the regions under which credit is better than money and vice versa. When the marginal cost to enhance enforcement is low, it is optimal to use credit and optimal credit limit decreases as such marginal cost increases. When such cost is very high, money dominates; in particular, agents would not even be willing to access the credit given the choice as it requires a high credit fee to finance the credit system. In contrast, the optimal arrangement in the intermediate range of such marginal costs is more subtle. In particular, in such a range, agents in a pure currency economy would be willing to use credit as it gives higher liquidity than money and the corresponding credit fee is not too high to offset the gain from credit. This implies that the private sector would prefer the credit system. However, there is a range where it is better for the government to not provide credit arrangement. The intuition here is that agents prefer credit because of the opportunity cost to hold money, but that cost that does not enter social welfare. As a result, when the liquidity provided by money is not too far from the credit economy, a pure currency economy in fact delivers higher welfare than a credit economy.

Related literature

This paper is related to a growing literature that studies endogenous credit limits by taking limited commitment and enforcement seriously. Conceptually, our formulation of endogenous credit limits follows that in Kehoe and Levine (1993) and Kocherlakota and Wallace (1998), and is similar to those taking the same inheritance to the Lagos-Wright (2005) framework for tractability, including Sanches and Williamson (2010), Chiu et al. (2018), and Bethune et al. (2018), among others.

The paper is also related to the literature that discusses the optimal use of means-ofpayments that follows Kocherlakota (1998). We have a similar model setup as in Gu et al. (2016), but in their paper the record-keeping technology is complete and costless. Their findings suggest that credit is neutral in terms of both allocation and welfare when money is valued, which is a special case of our results by letting the marginal cost of enforcement as zero. Other papers includes Bethun et al. (2018) whose focus is more on the non-stationary credit equilibrium that is welfare-improving, and Araujo and Hu (2018) in which the access to money and credit is exogenous.

Other papers emphasize the imperfection of record-keeping technology. In Kocherlakota and Wallace (1998) and Bethune et al. (2015), the record-keeping technology is incomplete and it has been shown that better quality of record-keeping always improves credit condition as it is costless. Due to the same reason, more credit is always welfare improving. Still others have introduced an exogenous costs to access credit but they assume perfect commitment therefore credit is unconstrained, as in Wang et al. (2020), Dong and Huangfu (2021). In contrast to these papers, we characterize both the incompleteness and the cost of recordkeeping technology and endogenize it by taking the limited-commitment friction seriously.

Another related paper is Lotz and Zhang (2016) and in their papers, the cost of accessing credit is paid by sellers. It leads to complementarities between buyers and sellers and hence multiple equilibria issue, moreover, it generates inefficiency from bargaining related holdup problems. Instead, we let buyers to pay the credit cost and have all the bargaining power, by which we have more tractable equilibrium analysis and can focus more on endogenizing the record-keeping technology and welfare analysis.

Earlier papers indicating that credit is not always welfare improving includes Chiu et al. (2018). They capture a general equilibrium effect of better access to credit as it increases the consumption of credit user, but also drives up the equilibrium price, which harms the money users. In our paper, agents endogenously choose the means-of-payment and the negative effect of credit comes from the cost of enforcement and limited commitment friction.

2 Environment

The baseline model is based on Lagos and Wright (2005). Time is discrete and has an infinite horizon. The economy is populated by two continuum sets of agents, labelled as *buyers* and *sellers*. Every period is divided into two stages. In the first stage there is a decentralized market (DM) where buyers and sellers meet in pairwise meetings, and only buyers desire to consume and only sellers can produce the DM good. The probability that a buyer meets a seller (and vice versa) is $\sigma \in (0, 1]$ and the buyer makes take-it-or-leave-it-offer to the seller. In the second stage there is a centralized market where all agents meet. All agents can consume and produce a single good, called the CM good, with linear preferences and a linear technology to produce. Both DM goods and CM goods are perfectly divisible and nonstorable.

The instantaneous utility functions of buyers and sellers are given by

$$U^{b}(y,x) = u(y) + x$$
 and $U^{s}(y,x) = -y + x$,

respectively, where y is the amount of DM consumption (production) and x is the amount of CM consumption (negative number is interpreted as production). The utility function is twice continuously differentiable with $u'(\cdot) > 0$, $u''(\cdot) < 0$, $u'(0) = +\infty$, $u'(\infty) = 0$ and u(0) = 0. Let y^* denote the consumption level that maximizes the match surplus u(y) - y, which is determined by $u'(y^*) = 1$. The cost function for the seller is assumed to be linear for expositional simplicity, and it has no bearing on our results. The discount factor is β , and denote the discount rate as $r = 1/\beta - 1$.

The government operates a financial system with a costly and imperfect monitoring technology. The buyer can access the financial system in the DM by incurring a credit fee χ (in terms of the CM goods) that is due in the CM.³ The monitoring technology records the buyer's credit transaction in the DM, payment of the credit fee and repayment of debt in the coming CM. The technology updates the buyer's status to be either a good standing (g) or bad standing (b), depending on the buyer's payments. The status of a buyer is observable to all sellers, but the status is updated imperfectly. If a buyer of status g who incurs debt d in the DM repays at least min $\{D, d\}$ to the seller and the credit fee χ to the government in the CM, he remains in status g. If he fails to repay min $\{D, d\}$ to the seller or the credit fee χ in the CM, his status will be updated to b with probability $\epsilon \in [0, 1]$.⁴

The probability of successful record-updating ϵ measures the enforcement level, or the quality of record-keeping.⁵ It is endogenously determined and is costly to implement. In particular, the government needs to spend $\xi v(\epsilon)$ units of CM good (in per buyer terms) to sustain ϵ and $v(\cdot)$ is a convex cost function. We start with a linear cost function $\xi v(\epsilon) = \xi \epsilon$ and generalize the main results under any convex cost function in the last section. With linear cost function, the parameter ξ is the marginal cost of monitoring and enforcement. Such cost would depend on the technology to review transactions and update records, but it also depends on the difficulty to overcome rent-seeking opportunities that would allow a defaulter to avoid a bad record. The government uses the credit fee, imposed on buyers who use credit, to finance this cost.⁶ As mentioned above, the credit fee is denoted by χ , and it

 $^{^{3}}$ Of course we can let sellers to pay the cost. But it will cause holdup problem and complementary, as discussed in Lotz and Zhang (2016).

⁴Alternatively, we can let buyers pay the credit fee ex ante before accessing credit, by doing so the recordupdating has to be done also ex ante, which means the technology labels a buyer who doesn't pay the fee directly as *bad*, before the buyer enters DM and uses credit. Because otherwise, buyers could always avoid the tax without leaving any record and being punished. The two versions, paying credit fee ex post or ex ante, do not influence the incentive constraint and the main results.

⁵This formulation is commonly used in literature to measure the quality of the record-updating, but it has other interpretations. Kocherlakota and Wallace (1998) interpret the probability as a measure of the average lag between transaction updating. In Bethune et al. (2015), it is a measure of the sophistication of the financial system. In Sanches and Williamson (2010), it represents the fraction of sellers that have monitoring potential.

⁶Of course, this can be done through a private sector as well; the main point here is that the monitoring is costly and there is a need for budget-balancing. We assume that this is done by the benevolent government

occurs when the buyer issues credit in a meeting, but is payable only in the following CM.⁷

There is only one asset, fiat money, which is intrinsically useless but is perfectly durable and recognizable. Let M_t denote the aggregate money supply at the beginning of period t, and in the first period, M_0 is given to the buyers evenly. In the baseline model we assume money supply is constant over time, and we discuss the case when money grows at a constant rate in the last section. There is a competitive market for money and CM goods in the CM and we express real balances as z (in terms of the CM goods). We consider stationary equilibrium where the aggregate real balance is constant over time.

In a DM meeting, a buyer can always use cash to finance his consumption. Or he can choose to issue credit but subject to the credit fee and repayment constraint. If a buyer default on his obligation and being detected and labeled as *bad*, the punishment is permanent autarky. In literature, the more commonly used punishment of being caught is permanently exclusion from the credit system. Our punishment may be too strict as a *bad* buyer could always use money if money has value in the equilibrium. We adopt this punishment as it simplify the incentive constraint and does not affect the welfare ranking when money and credit coexist. Moreover, it does not affect any analysis in the pure credit equilibrium as money has no valued and being excluded from using credit implies autarky. ⁸

to avoid any further agency problem.

⁷We following Araujo and Hu (2018) and interpret the cost of credit as the social resources invested to the record-keeping system, including to operate the credit card system and to screen, monitor and reveal information. This social cost is financed by taxation called credit fee and agents can always default on it. There are other ways to model costly credit. For example, in Wang et al. (2020) and Dong and Huangfu (2021), buyers incur a fixed or proportional utility cost in the decentralized market while using credit. In Lotz and Zhang (2016), sellers randomly incur heterogeneous utility cost and the cost is zero for a fraction of sellers. Such utility cost allows for heterogeneity but avoid the payment issue that could indirectly impact the incentive constraint and therefore credit limit, which is our main focus in this paper.

⁸A micro-foundation explanation of such punishment is that the trading protocol depends on buyer's status, used in Araujo and Hu (2018). Specifically, a buyer with a good standing makes a take-it-or-leave-it offer to the seller, but the seller makes a take-it-or-leave-it to a buyer with a bad standing. They show that such punishment in fact a feature of the optimal trading protocol to enhance incentive-compatibility.

3 Equilibrium analysis

In this section, we begin with a partial equilibrium analysis for the endogenously given credit limit D and credit fee χ . We characterize different stationary equilibrium according to the use of different means-of-payment.

In particular, we show that a monetary equilibrium (z > 0) exists only when the credit limit is not too high, and there is a threshold for the credit limit above which money is dominated by credit and a pure credit equilibrium exists. When credit limit is below that threshold, depending on the value of credit limit and credit fee, either a coexistence equilibrium exists where buyers use both money and credit in the decentralized trade, or a pure monetary equilibrium exists where only money is used even though credit is provided. For the non-monetary equilibrium (z = 0), we show that there is another threshold of credit limit above which buyers access credit and a pure credit equilibrium exists, otherwise buyers do not trade in the decentralized market even though credit is provided, and the economy is in an autarky equilibrium.

In section 3.1, we fully characterize the range of (D, χ) such that different equilibrium exists. Moreover, we show that when both monetary and non-monetary equilibrium exist, the buyers' ex ante payoff in the monetary equilibrium is always higher than in the non-monetary equilibrium. Later on we will show that the social welfare in the monetary equilibrium is also higher than in the non-monetary equilibrium when both equilibrium exist.

In section 3.2, we move on to endogenize the credit limit. For given monitoring technology ϵ and credit fee χ , we find the set of credit limit that satisfies the buyers' incentives to repay and government budget balancedness. Then we discuss the relationship between the highest implementable credit limit and the environment rate.

3.1 Equilibrium under a given credit limit

Here we assume an exogenously given credit limit $D \leq y^*$, and we assume that buyers always repay their debts and credit fees in the CM if they choose to access the financial system in the DM trade. In the last subsection we shall consider the incentive problem of repayment and endogenize this limit D.

For a given D and assuming that buyers always repay their debts up to D, a stationary equilibrium consists of equilibrium real balance, z, that each buyer holds leaving the CM and equilibrium DM trade, (y, p, d), where $p \leq z$ is the equilibrium money transfer (denominated in CM-goods terms) from the buyer to the seller, $d \leq D$ is the equilibrium credit (denominated in CM-goods terms) that each buyer issues to seller and y is the equilibrium output. Moreover, equilibrium requires z is the optimal money holding for each buyer, and, whenever d > 0, buyers are willing to access the financial system in the DM and repay the debt d and pay the credit fee χ . Finally, sellers are willing to produce the amount y in exchange of the payment p + d.

In subsection 3.1.1 we discuss the range of given (D, χ) such that a monetary equilibrium exists (z > 0), and if buyers do not issue credit d = 0, we call it a pure monetary equilibrium. Otherwise, if buyers use credit d > 0, we call it a coexistence equilibrium. In section 3.1.2, we move on to the non-monetary equilibrium where z = 0. If buyers access the financial system d > 0, we call it a pure credit equilibrium, otherwise when d = 0, we call it an autarky equilibrium.

3.1.1 Monetary equilibrium

We begin with monetary equilibrium and consider the buyer's CM problem. To do so, denote the value function for a buyer who did not access the financial system in the previous DM and enters CM with z real balances by $W^m(z)$, and denote a buyer who accessed the financial system in DM and enters CM with z real balance and d debt by $W^c(z, d)$. We assume that sellers do not carry money across periods and sell all their money holding accumulated from the DM in the coming CM, an assumption that is with no loss of generality, and hence only consider buyers' problems.

The standard Lagos-Wright argument implies that the CM value function is linear in z and d, that is,

$$W^m(z) = z + W^m(0)$$
 and $W^c(z, d) = z - d - \chi + W^m(0)$. (1)

Note that $W^c(0,0) = W^m(0) - \chi$, which reflects the fact that the buyer is obligated to pay the credit fee χ , independent of his debt issuance. Note also that $W^m(0)$ solves the CM problem for a buyer who enters the CM with no money and did not access the financial system, and, by linearity, this represents the typical CM problem for the buyer, regardless whether he has accessed the financial system in the previous DM.

Now we move to the DM problem. Denote the DM value function for a buyer with real balance z by V(z), and it satisfies

$$V(z) = \sigma \max\{V^{m}(z), V^{c}(z)\} + (1 - \sigma)W^{m}(z),$$
(2)

where $V^{m}(z)$ is the continuation value without accessing the financial system and solves

$$V^{m}(z) = \max_{y^{m} \leqslant p, p \leqslant z} \left\{ u(y^{m}) + W^{m}(z-p) \right\},$$
(3)

in which (y^m, p) is the offer made to the seller, subject to the liquidity constraint $p \leq z$, and the seller participation constraint $y^m \leq p$; and where $V^c(z)$ is the continuation value by accessing the financial system and solves

$$V^{c}(z) = \max_{y^{c} \leqslant p+d, p \leqslant z, d \leqslant D} \left\{ u(y^{c}) + W^{c}(z-p,d) \right\},$$
(4)

in which (y^c, p, d) is the offer made to the seller, subject to the liquidity constraint $p \leq z$, the credit limit $d \leq D$ and the seller participation constraint $y^c \leq p + d$. If a buyer is not matched with a seller, the buyer takes real balance z to the CM without debt or tax obligation.

By (1) and (3), the consumption of a buyer who only uses money in the DM is determined by $y^m = y^m(z) = y^*$ if $z \ge y^*$, and $y^m = y^m(z) = z$ otherwise. Similarly, by (1) and (4), the consumption of a buyer who accesses the financial system in DM is determined by $y^c = y^c(z, D) = y^*$ if $z + D \ge y^*$, and $y^c = y^c(z, D) = z + D$ otherwise. In both cases, buyers will use all available payment capacity in DM to consume, until the first-best level of output is achieved.

Now consider the CM problem, which is given by

$$W^{m}(0) = \max_{z} \left\{ -z + \beta V(z) \right\}.$$
 (5)

When choosing his money holding, the buyer anticipates his DM decision to access the financial system or not (which is embedded in the Bellman equation for V in (2)), and plans accordingly. To characterize the optimal decision, define

$$\Psi^{m} \equiv \max_{z} \left\{ -rz + \sigma[u(y^{m}(z)) - y^{m}(z)] \right\},$$
(6)

whose solution is denoted by \bar{z} and it solves $u'(\bar{z}) = \frac{\sigma + r}{\sigma}$, and define

$$\Psi^{c}(D) \equiv \max_{z} \left\{ -rz + \sigma[u(y^{c}(z+D)) - y^{c}(z+D) - \chi] \right\},$$
(7)

with solution z = 0 if $D > \overline{z}$; $z = \overline{z} - D$ otherwise. The following lemma shows that the CM problem can be solved by comparing Ψ^m and $\Psi^c(D)$, and it summarizes equilibrium allocations and means-of-payments as well.

Lemma 3.1 Let $\bar{\chi} \equiv \frac{\sigma[u(y^*)-y^*]-\Psi^m}{\sigma}$, we characterize the monetary equilibrium with the highest z below, in which y = z + d.

- (1) If $\chi > \overline{\chi}$, then in equilibrium real balance $z = \overline{z}$ and buyers do not use credit in the DM.
- (2) Otherwise, there exists $\tilde{D} = \tilde{D}(\chi)$, determined by $\Psi^m = \Psi^c(\tilde{D})$ such that

(2.1) if $D < \tilde{D}$, in equilibrium $z = \bar{z}$ and buyers do not use credit in the DM; (2.2) if $D \in [\tilde{D}, \max\{\bar{z}, \tilde{D}\})$, in equilibrium $z = \bar{z} - D$ and d = D in the DM; (2.3) if $D \ge \max\{\bar{z}, \tilde{D}\}$, there is no monetary equilibrium.

Intuitively, the buyer's decision to access the financial system depends on the potential gain from trade achievable, which depends on the credit limit, D, and the cost of accessing it, the credit fee χ . Since the benefit is bounded above by the first-best gain from trade, a very high fee for credit trades will deter the buyer from accessing the financial system. Lemma 3.1(1) characterizes the precise upper bound $\bar{\chi}$, above which buyers do not access the financial system, regardless of the credit limit. In this case, there is a pure monetary equilibrium and the equilibrium real balance is $z = \bar{z}$.

When χ is below that threshold, Lemma 3.1(2) shows that buyers' access to the financial system will depend on the credit limit. The critical value is given by the threshold \tilde{D} , which is determined by $\Psi^c(D) = \Psi^m$, and hence, this result also shows that the buyer solves max { $\Psi^c(D)$, Ψ^m } to determine whether to use credit or not. When the credit limit is below \tilde{D} (which is a function of χ), Lemma 3.1 (2.1) states that the buyer will not issue credit but use money alone and there is a pure monetary equilibrium. Otherwise, Lemma 3.1 (2.2) states that the buyer will use credit, and supplement it with cash if the credit limit is not high enough , and there exists a coexistence equilibrium. Finally, when $D \ge \max{\{\tilde{D}, \bar{z}\}}$, there is no monetary equilibrium since using credit gives the buyer higher interim payoff than using money.

3.1.2 Non-monetary equilibrium

When money has no value, the buyer's problem follows same analysis as in subsection 3.1.1 but with z = 0. Equation (6) and (7) can be rewritten as $\Psi^m = 0$ and $\Psi^c(D) = \sigma[u(D) - D - \chi]$. For given credit limit $D \leq y^*$ and credit fee χ , the next lemma describes buyers decision to access the financial system and the stationary equilibrium allocations.



Figure 1: Stationary equilibrium for given credit limit and credit fee

Lemma 3.2 Let $\hat{\chi} \equiv u(y^*) - y^*$. A non-monetary equilibrium always exist with z = 0, and we characterize the stationary equilibrium below with y = d.

- (1) If $\chi > \hat{\chi}$, buyers do not use credit in DM with d = 0 and the economy is in autarky.
- (2) Otherwise, there exist $\hat{D}(\chi)$, determined by $\sigma[u(\hat{D}) \hat{D} \chi] = 0$, such that
 - (2.1) if $D \ge \hat{D}(\chi)$, in equilibrium buyers only use credit d = D in the DM;
 - (2.2) if $D < \hat{D}(\chi)$, d = 0 in the DM and the economy is in autarky.

In Lemma 3.2, when the credit fee is higher than the first-best gain from trade $(\chi > \hat{\chi})$, buyers do not access the financial system for any credit limit. As money has no value, the DM trade is not active.

When the credit fee is lower than that threshold, buyer's decision to access the financial system depends on the credit limit. In Lemma 3.2 (2.1) when the credit limit is higher than a threshold \hat{D} , buyers use credit to trade in the DM, and a pure credit equilibrium exists. Otherwise in Lemma 3.2 (2.2) when the credit limit is low, the gain from trade cannot compensate the cost of using credit, so buyers do not access the financial system and do not trade in the DM, regardless of the credit limit.

We summarize the results in Lemma 3.1 and Lemma 3.2 in Figure 1. The red area $(D < \max\{\bar{z}, \tilde{D}\})$ represents the range such that a pure monetary equilibrium (PM) or a coexistence equilibrium (MC) exists. The gray area $(D \ge \hat{D})$ represents the range such that

a pure credit equilibrium (PC) exists. Note that for given credit limit, when the credit fee is low, a pure credit equilibrium exists and when the cost goes up, a pure monetary equilibrium exists. There is a range of credit fee such that a money and credit equilibrium coexists with pure credit equilibrium, a finding similar as in Wang et al. (2020) under fixed cost, except that the credit is unconstrained in their model due to perfect commitment, which could be seen as a special case where D approaches y^* .

The next lemma compares the buyers' ex ante payoff, which is also the equilibrium social welfare, when multiple equilibria exist. Note that, however, a full discussion of welfare requires government budget balancedness. Nevertheless, the following results hold for any χ , and hence can be easily transformed to the case where we introduce government budget constraint.

Lemma 3.3 Multiple equilibria exist when $D \in [\hat{D}, \max\{\bar{z}, \tilde{D}\})$, and:

- (1) When $D \in [\tilde{D}, \bar{z})$, the buyer's ex ante payoff in the coexistence equilibrium, which is $\sigma[u(\bar{z}) \bar{z} \chi]$, is higher than in a pure credit equilibrium, which is $\sigma[u(D) D \chi]$;
- (2) When $D \in [\hat{D}, \tilde{D})$, the buyer's ex ante payoff in the pure monetary equilibrium, which is $\sigma[u(\bar{z}) \bar{z}]$, is higher than in a pure credit equilibrium, which is $\sigma[u(D) D \chi]$.

According to Lemma 3.3 (1), there exist a coexistence equilibrium and a pure credit equilibrium, and in both cases, buyers access the financial system and pay credit fee. But since the credit limit is lower than the amount of real balances in a pure monetary equilibrium, the buyer could use money to compensate the lack of credit and achieve higher trade surplus in the coexistence equilibrium. In Lemma 3.3 (2), there exist a pure monetary equilibrium and a pure credit equilibrium. Because the credit limit is lower than the threshold \tilde{D} , Lemma 3.1 shows that the buyer's value of using credit is lower than using money alone, which is $\Psi^c(D) < \Psi^m$, and therefore, the buyer's ex ante payoff is also higher in the pure monetary equilibrium.

Before we move on to endogenize the credit limit and credit fee, it is useful to define the set of credit limit and select the stationary equilibrium that we will focus on. Lemma 3.3 indicates that when $D \in [\hat{D}, \max\{\bar{z}, \tilde{D}\})$, a pure credit equilibrium is never socially optimal. Therefore we do not consider them in the following analysis. We denote the credit limit that is of interest as $D \in [\tilde{D}, y^*]$, where when $D < \bar{z}$, we focus on the coexistence equilibrium, as described in Figure 2.



Figure 2: The set of credit limit $D \in [\tilde{D}, y^*]$

3.2 Endogenous credit limit and incentive compatibility

Here we endogenize the credit limit by considering the buyer's incentive to repay, taking ϵ (the monitoring technology) and χ as given. We analyze equilibrium conditions taking buyers' repayment decisions into account, as well as government budget constraint to finance the cost of the monitoring technology ϵ . In particular, for a candidate credit limit D and enforcement level ϵ , the pair is said to be *implementable* if buyers are willing to access the financial system and repay up to D plus the credit fee, and if the credit fees collected are sufficient to finance the cost associated with ϵ , given the equilibrium allocation characterized by Lemma 3.1 and Lemma 3.2.

In the previous section we have characterized the optimal behaviour for a buyer with good standing. Note that since for a buyer with bad standing, it is the seller who makes the TIOLI offer, and since the cost of holding money across periods is positive, his continuation value is zero.

For incentive compatibility, consider a buyer with good standing in the CM, with a loan equal to the credit limit $D \leq y^*$ from the previous DM and the credit fee χ . If the

buyer repays, his continuation value is $W^c(z, D) = z - D - \chi + W^m(0)$. Alternatively, the buyer could default. Since the record updating technology does not depend on the size of the default up to the credit limit plus the credit fee, it is optimal to default on the total obligation if the buyer would choose to default any amount. By doing so, with probability ϵ his standing will be changed to bad, and, as we have seen, it implies a zero continuation value. Thus, to ensure that such a buyer repays his obligation, we need

$$W^c(z,D) \ge z + (1-\epsilon)W^m(0),$$

where the left-side is the continuation value if he repays, and the right-side is the continuation value otherwise: z is from selling his money holding and with probability $1 - \epsilon$ he is not caught and can continue as a buyer with good standing. As mentioned, we focus on the case where $D \ge \tilde{D}(\chi)$, hence $\Psi^c(D) \ge \Psi^m$, and we can rewrite the condition as

$$r(D+\chi) \leqslant \epsilon \Psi^c(D). \tag{8}$$

The government uses the credit fees to finance the cost of monitoring, and only buyers who access the financial system pay those fees. Assume that all buyers with the access use the system, the government faces the following budget constraint:

$$\sigma\chi = \xi\epsilon. \tag{9}$$

In principle, budget constraint would only require the left-side of (9) to be no less than the right-side; we only consider equality because it is welfare-maximizing and simplifies the notation.

To summarize, the pair, (D, ϵ) , is implementable if, for χ satisfying (9), the incentive constraint (8) holds, and if the credit equilibrium exists, as in Lemma 3.1 and Lemma 3.2. Now, as we focus on the set $D \in [\tilde{D}(\chi), y^*]$, by (9) we can express \tilde{D} as a function of (ϵ, ξ) and hence we use the notation $\tilde{D}(\epsilon; \xi)$ from now on. The following lemma characterizes implementable (D, ϵ) that $D \in [\tilde{D}, y^*]$. **Lemma 3.4** For a given marginal cost of enforcement, ξ , the pair (D, ϵ) is implementable with $D \ge \tilde{D}(\frac{\xi\epsilon}{\sigma}) \equiv \tilde{D}(\epsilon;\xi)$ if and only if

$$\begin{cases} \Psi^m + rD - \xi \epsilon \geqslant \frac{r}{\epsilon}D + \frac{r\xi}{\sigma} & \text{if } D \in [\tilde{D}, \max\{\bar{z}, \tilde{D}\}); \\ \sigma[u(D) - D] - \xi \epsilon \geqslant \frac{r}{\epsilon}D + \frac{r\xi}{\sigma} & \text{if } D \in [\max\{\bar{z}, \tilde{D}\}, y^*); \\ \sigma[u(y^*) - y^*] - \xi \epsilon \geqslant \frac{r}{\epsilon}D + \frac{r\xi}{\sigma} & \text{if } D = y^*. \end{cases}$$
(10)

Moreover, for each pair (ϵ, ξ) there is a threshold $\overline{D} = \overline{D}(\epsilon; \xi)$ such that D satisfies (10) if and only if $D \in [\tilde{D}, \overline{D}]$ (which might be empty in case $\tilde{D} > \overline{D}$).

Given Lemma 3.4, we can then fully characterize implementable (D, ϵ) when $D \in [\tilde{D}, y^*]$.

- **Proposition 3.1** (1) For all $\xi > \frac{\sigma}{\sigma+r}\Psi^m$, there is no implementable (D, ϵ) such that $D \ge \tilde{D}(\epsilon; \xi)$.
 - (2) For all $\xi \leq \frac{\sigma}{\sigma+r} \Psi^m$, there exists a threshold $\bar{\epsilon}(\xi) > 0$ such that for any $\epsilon \in [0, \bar{\epsilon}(\xi)]$, $\tilde{D}(\epsilon; \xi) \leq \bar{D}(\epsilon; \xi)$. Moreover, in this range,
 - $\begin{array}{l} (2.a) \quad \frac{\mathrm{d}\bar{D}(\epsilon;\xi)}{\mathrm{d}\epsilon} \ge 0 \ and \ \frac{\mathrm{d}\bar{D}(\epsilon;\xi)}{\mathrm{d}\xi} \leqslant 0, \ with \ strict \ inequality \ if \ \bar{D} < y^*; \\ (2.b) \ \bar{D}(0;\xi) = 0, \ \bar{D}(\bar{\epsilon}(\xi);\xi) \ge \bar{z} \ and \ the \ inequality \ is \ strict \ for \ all \ \xi < \frac{\sigma}{\sigma+r} \Psi^m. \end{array}$

Proposition 3.1 fully characterizes implementable (D, ϵ) when $D \in [D, y^*]$. Part (1) shows that when the marginal cost of enforcement is too high $(\xi > \frac{\sigma}{\sigma+r}\Psi^m)$, there is no implementable (D, ϵ) such that $D \ge \tilde{D}$. Otherwise, for any enforcement rate ϵ below the threshold $\bar{\epsilon}$, Lemma 3.4 implies that any $D \in [D, \bar{D}]$ is implementable under ϵ and Proposition 3.1 (2) implies that that range is not empty. As typical in the credit economies with endogenous credit limits, this implies that there is a continuum of credit limits that are incentive compatible when ξ is low.⁹ Here we show more: there is also a range of enforcement rates that are both incentive feasible and fiscally feasible as enforcement is costly.

⁹See, for example, Bethune et al. (2018) for a general discussion on this point.

Our main focus, however, is on optimal arrangements that maximize the social welfare. For any given $\xi \leq \frac{\sigma}{\sigma+r} \Psi^m$ and $\epsilon \leq \bar{\epsilon}(\xi)$, as we will see later, the highest implementable credit limit is $\bar{D} = \bar{D}(\epsilon; \xi)$. Proposition 3.1 (2.a) then shows that \bar{D} is an increasing function of ϵ for any $\epsilon \in [0, \bar{\epsilon}(\xi)]$. Intuitively, an increasing in the enforcement rate ϵ influences the buyer's repayment decision through two effects. The first effect is an increase in the probability of being caught and losing all the trade surplus, which encourages buyers to repay. This then allows for a higher credit limit, which also encourages buyers to access the financial system. However, the second effect is an increase in the credit fee, χ , which makes accessing the financial system more expensive and makes it more tempting to default as the credit fee is also part of the buyer obligation. It turns out that the first effect is stronger, and the reason is that, to make the arrangement incentive compatible, the marginal cost ξ cannot be too high to begin with and this limits the magnitude of the second effect. In contrast, an increase in ξ (but holding ϵ at constant) only has the second effect, and it decreases \bar{D} .

Proposition 3.1 (2.b) describes two boundary conditions of $\overline{D}(\epsilon; \xi)$. Clearly, when $\epsilon = 0$, buyers face no penalty by defaulting, and, anticipating this, sellers would never issue any credit, $\overline{D}(0;\xi) = 0$. When the enforcement is the highest, $\epsilon = \overline{\epsilon}(\xi)$, the credit limit exceeds \overline{z} and hence the buyer does not carry cash. Since \overline{D} is continuous in ϵ , this also implies that there is a cutoff point $\underline{\epsilon}(\xi)$ below which the credit limit \overline{D} is low and hence the buyer carries cash to compliment the difference between \overline{z} and \overline{D} , but above which the buyer only uses credit and does not carry cash. See Figure 3 for a depiction of the relationship between \overline{D} and ϵ , and the ranges for optimal portfolio of the buyer.

4 Welfare Analysis and Optimal Policy

Here we study the optimal scheme from a social planner's perspective, who takes ξ as given and chooses ϵ and D, subject to implementability. We do this in two steps. First, for a given ξ , we trace the welfare change as D varies and show that it is not necessarily optimal to implement the highest possible credit limit. This is also related to the non-monotonicity noted in the introduction. Second, we show that the optimal credit limit decreases with ξ ,



Figure 3: The highest implementable credit limit \overline{D} for a given enforcement rate ϵ

and it collapses to zero for higher ξ 's even though a pure credit equilibrium is implementable.

We first define welfare. A stationary allocation in our economy consists of only the enforcement rate, ϵ , and the level of DM trade, y. Given the allocation, (ϵ, y) , the corresponding welfare is

$$\mathcal{W}(\epsilon, y) = \sigma[u(y) - y] - \xi\epsilon.$$
(11)

In the following analysis we also maintain budget-balancedness according to (9).

4.1 Social welfare in credit equilibrium

To discuss welfare in arrangements where credit is used, we first consider the alternative arrangement where credit is not used at all. If the social planner decides not to use credit, it is then optimal to choose $\epsilon = 0$. In that case, the welfare is given by

$$\mathcal{W}^m = \sigma[u(\bar{z}) - \bar{z}],\tag{12}$$

where \bar{z} solves (6), the equilibrium real balance holding in a pure monetary equilibrium where credit is not used. Since the pure monetary equilibrium is implementable for any ξ by setting $\epsilon = 0$, \mathcal{W}^m serves as a lower bound to the optimal welfare. Moreover, it will not be optimal to use credit unless it can achieve a higher welfare than \mathcal{W}^m . Finally, since a credit equilibrium is implementable with $D \ge \tilde{D}$ only if $\xi \le \frac{\sigma}{\sigma+r}\Psi^m$, we only need to consider that range.

For $\xi \leq \frac{\sigma}{\sigma+r} \Psi^m$, from Proposition 3.1, a pair (ϵ, D) is incentive compatible if and only if $\epsilon \leq \bar{\epsilon}(\xi)$ and $D \in [\tilde{D}(\epsilon; \xi), \bar{D}(\epsilon; \xi)]$. Given an implementable pair, the equilibrium DM trade is given by $y^c = y^c(z+D)$ with z the solution to (7), that is, $y^c = \max\{\bar{z}, D\}$. Thus, to maximize welfare among credit equilibria, the problem becomes

$$\max_{(D,\epsilon)} \mathcal{W}^{c}(D,\epsilon) \equiv \sigma \left[u \left(\max\{\bar{z}, D\} \right) - \max\{\bar{z}, D\} \right] - \epsilon \xi,$$
(13)
s.t. $\tilde{D}(\epsilon;\xi) \leq D \leq \bar{D}(\epsilon;\xi).$

Now, since y^c increases in D and the welfare increases in y^c (note that D and hence y^c is always below or equal to y^*), it is optimal to choose $D = \overline{D}$ for any given ϵ . As a result, we can reduce the problem to the choice of ϵ . However, Proposition 3.1 (2.a) implies that there is a one-to-one relationship between \overline{D} and ϵ as long as $\overline{D} < y^*$. Now, once \overline{D} reaches y^* , a higher ϵ does not increase the first term in (13) but only decreases the total welfare by increasing the cost in the second term. Thus, we may restrict our choice of ϵ so that there is always one-for-one correspondence between \overline{D} and ϵ (so, if $\overline{D} = y^*$ is considered, we just choose the lowest ϵ so that $\overline{D}(\epsilon; \xi) = y^*$).

Now, let $\overline{D}(\xi) \equiv \overline{D}(\overline{\epsilon}(\xi);\xi)$, the highest \overline{D} implementable under the marginal cost of enforcement ξ . We can then define the inverse of $\overline{D}(\cdot;\xi)$ for the given ξ , and take $\epsilon(D;\xi)$ to be the smallest ϵ so that $\overline{D}(\epsilon;\xi) = D$. Note that the function $\epsilon(D;\xi)$ in $[0,\overline{D}]$ is well-defined and continuous as \overline{D} is strictly increasing up to $\overline{D} = y^*$ by Proposition 3.1 (2.a) and (2.b). To solve (13), it is then further reduced to

$$\max_{D \in [0, \bar{D}(\xi)]} \mathcal{W}^c[D, \epsilon(D; \xi)].$$
(14)

In this maximization problem, implicitly any choice of D is already the optimal one for the enforcement level $\epsilon(D;\xi)$. Since $\mathcal{W}^c(D,\epsilon)$ is increasing in D and decreasing in ϵ , the program (14) highlights the essential trade-off involved in the optimal credit limit when the level of enforcement is endogenous: a higher D increases the trade surplus in the first term in (13), but it also requires a higher ϵ that increases the cost in the second term. The next proposition shows that this is a non-trivial trade-off.

Proposition 4.1 Let $\xi \leq \frac{\sigma}{\sigma+r} \Psi^m$ be given.

- (1) For any $D < \overline{z}$, $\mathcal{W}^c[D, \epsilon(D; \xi)]$ strictly decreases with D.
- (2) Suppose that $u(y) = \frac{y^{\alpha}}{\alpha}$ with $\alpha \in (0,1)$. If $r < \sigma \frac{1-\alpha}{\alpha}$, then $\mathcal{W}^c[D, \epsilon(D; \xi)]$ first increases with D but then decreases with D for $D \in [\bar{z}, \bar{D}]$.

Proposition 4.1 shows a non-monotonic relationship between credit limits and social welfare. According to Proposition 4.1 (1), it is never optimal to choose a $D < \bar{z}$. Indeed, for implementable D in that range, in equilibrium the buyer uses both money and credit, but equilibrium DM trade is not affected by the credit limit and stays at \bar{z} as D increases. However, higher D still requires higher ϵ and hence welfare strictly decreases with D. For higher D's, Proposition 4.1 (2) gives a sufficient condition for the welfare to be non-monotonic in D. Indeed, a straightforward differentiation yields

$$\frac{\mathrm{d}\mathcal{W}^c}{\mathrm{d}D} = \sigma[u'(D) - 1] - \xi \frac{\mathrm{d}\epsilon(D;\xi)}{\mathrm{d}D}.$$
(15)

Since the first term decreases with D because of the concavity of u, when D is relatively high, the second term dominates the first. The condition that r is relatively small ensures that this range exists.

We demonstrate this reserve U-shaped effect of increase in the credit limit in Figure (4), where r = 0.08, $\sigma = 0.25$, $\alpha = 0.75$ and $\xi = 0.005$. So the first-best trade outcome is $y^* = 1$ with the first-best social welfare $\mathcal{W}^* = 0.0833$, and the DM trade in a pure monetary equilibrium is $\bar{z} = 0.3293$ with welfare $\mathcal{W}^m = 0.0625$. In a pure credit equilibrium, the highest implementable pair $(D, \epsilon) = (0.958, 1)$. The curve represents the social welfare when credit is used. In the coexistence equilibrium, social welfare strictly decreases with D, as in Proposition (4.1)(1). In the pure credit equilibrium, social welfare first increases and then decreases with D, and reaches the highest when $(D, \epsilon) = (0.9208, 0.9596)$. The highest social welfare is 0.0783, accounting for 94.03% of the first-best social welfare, and improve 25.28% from a pure monetary social welfare.



Figure 4: The social welfare for given highest implementable credit limit

4.2 Cost of enforcement and optimal policy

Now we turn to the optimal arrangement, and study how changes in the marginal cost of enforcement, ξ , affect the optimal means-of-payments. First we give a proposition to characterize the optimal means-of-payment to be used.

Proposition 4.2 Let $\xi < \frac{\sigma}{\sigma+r}\Psi^m$ be given. Denote the optimal welfare in a credit equilibrium by $\mathcal{W}^c(\xi)$, then there exists $\hat{\xi} < \frac{\sigma}{\sigma+r}\Psi^m$ such that

$$\mathcal{W}^{c}(\xi) \ge \mathcal{W}^{m} \text{ if and only if } \xi \in [0, \hat{\xi}].$$
 (16)

Hence, it is optimal to implement a pure credit equilibrium for $\xi \in [0, \hat{\xi}]$ and to implement a pure monetary equilibrium for $\xi > \hat{\xi}$. Moreover, $\mathcal{W}^{c}(\xi)$ strictly deceases with ξ .

Proposition 4.2 shows that the solution to the optimal welfare under credit arrangement always exists. Moreover, it shows that there is a threshold for the marginal cost of enforcement, $\hat{\xi}$, below which a pure credit equilibrium is optimal and above which a pure monetary equilibrium is optimal. Note that above the threshold $\hat{\xi}$ a pure credit equilibrium is implementable but it is dominated by a pure monetary equilibrium. The intuition for this result is simple. When ξ is close to the threshold $\frac{\sigma}{\sigma+r}\Psi^m$, although a pure credit equilibrium is implementable, the maximal credit limit D is close to \bar{z} and hence the trading surplus in credit trade is not too much better than the monetary trade in a pure monetary equilibrium. However, the credit economy requires a cost of enforcement that is not needed in a monetary equilibrium, and hence welfare in the credit economy must be lower. Our next result states that for ξ relatively small, the optimal credit limit decreases with the marginal cost of enforcement.

Proposition 4.3 There exists a threshold $\tilde{\xi} \in (0, \hat{\xi}]$ such that for all $\xi < \tilde{\xi}$, the optimal credit limit, denoted by $D^* = D^*(\xi)$, decreases with ξ . If $u(y) = \frac{y^{\alpha}}{\alpha}$ with $\alpha \in (0, 1)$, then $\tilde{\xi} = \hat{\xi}$ and D^* strictly decreases with ξ for all $\xi \in [0, \hat{\xi}]$.

Proposition 4.3 shows that the optimal credit limit decreases with ξ , at least for a range of small ξ 's, and, if u is CRRA with u(0) = 0, then this holds for all ξ 's under which it is optimal to use credit. In Figure 5 we depict the optimal credit limit, D^* , and the optimal social welfare, \mathcal{W} , both as functions of ξ . As can be seen from the right panel, around $\xi = \hat{\xi}$ there is a kink to the optimal welfare, which is resulted from the discontinuity in the amount of liquidity in the economy. Indeed, the left panel shows that for any $\xi < \hat{\xi}$, the optimal credit limit is bounded away from \bar{z} , but for $\xi > \hat{\xi}$ the real balance in the pure currency economy is constant at \bar{z} . As a result, the output level in the DM will have a discontinuous increase when the marginal cost of enforcement decreases below $\hat{\xi}$. Note that this last result does not depend on the functional form of the utility function u, as $D^*(\xi) > \bar{z}$ for all $\xi < \hat{\xi}$.



Figure 5: Optimal credit limit and social welfare

4.3 Convex cost of enforcement

We remark here that while we have assumed a linear enforcement cost, our results can be extended to a convex cost case, $\xi v(\epsilon)$ with $v'(\cdot) > 0$, $v''(\cdot) > 0$, v(0) = 0, v'(0) = 0 and v(1) = 1. The analysis then follows the same path as the linear cost case, and we only highlight a few key results. We will show that the monotonic relationship between the highest implementable credit limit and enforcement rate only holds when the cost parameter ξ is sufficiently small, and in this range, the welfare in pure credit equilibrium decreases with ξ . Interestingly, with convex cost, there can be a non-monotonic relationship between the cost parameter and the optimal enforcement level.

With government budget $\sigma \chi = \xi v(\epsilon)$, the incentive constraint, as equation (8), is given by,

$$r(D + \frac{\xi v(\epsilon)}{\sigma}) \leqslant \epsilon \Psi^c(D), \tag{17}$$

and the next proposition gives a sufficient condition for implementability:

Proposition 4.4 When $\xi \leq \min\{\frac{1}{v'(1)}\frac{\sigma}{\sigma+r}\Psi^m, r\bar{z}\}$, for any $\epsilon \in [0,1]$, $D \in [\tilde{D}(\epsilon;\xi), \bar{D}(\epsilon;\xi)]$ is implementable and is not empty, with $\frac{d\bar{D}(\epsilon;\xi)}{d\epsilon} \geq 0$.

Note that the influence of enforcement rate on the implementable credit limit depends on the magnitude of three opposite effects: on the right side of equation (17), higher enforcement rate increases the probability of being caught, which extensively increases the cost of default. But the continuation value of accessing credit decreases as credit is more expensive, which affects the intensive margin of default. On the left side, higher enforcement rate also increases the benefit from default, which in turn discourages buyers from using credit. In Proposition 4.4, we show that only when enforcement is not too costly (ξ is sufficiently small or $v'(\epsilon)$ is low), better enforcement leads to higher implementable credit limit.

Given ξ is sufficiently small, the optimal welfare and credit limit satisfies:

Proposition 4.5 Let $\xi \leq \min\{\frac{1}{v'(1)}\frac{\sigma}{\sigma+r}\Psi^m, r\bar{z}\}$, the optimal welfare in the pure credit equilibrium strictly decreases with ξ : $\frac{dW^c(\xi)}{d\xi} < 0$ with $W^c(0) > W^m$. Moreover, optimal credit



Figure 6: Optimal credit limit and enforcement rate

limit $D^*(\xi)$ decreases with ξ for ξ sufficiently small.

The proof of Proposition 4.5 is same as for Proposition 4.1 and 4.3. Interestingly, with convex cost, the optimal credit limit still decreases with the marginal cost when it is small, but there can be a non-monotonic relationship between the cost parameter ξ and the optimal enforcement level: $\epsilon^* = \epsilon(D^*(\xi); \xi)$:

$$\frac{d\epsilon(D^*(\xi);\xi)}{d\xi} = \frac{d\epsilon}{dD^*}\frac{dD^*}{d\xi} + \frac{d\epsilon}{d\xi},\tag{18}$$

where the first term is negative and captures an indirect effect of ξ on ϵ^* : when ξ increases, the optimal credit limit decreases, so the enforcement rate needed to implement a lower credit limit also decreases. However, the second term is positive and captures a direct effect: as ξ increases, credit fees increase and agents are less likely to use credit. So for a given credit limit, it requires a higher enforcement rate to implement it.

Therefore, the relationship between ϵ^* and ξ , and also ϵ^* and D^* can be non-monotonic in some cases. In Figure 6 we depict such an example. Here we have $u(y) = \frac{(y+b)^{\alpha}-b^{\alpha}}{\alpha}$ with $\alpha = 0.25, \sigma = 0.18, r = 0.15, b = 0.005$, and the cost of enforcement is given by $\xi \epsilon^m$ with m = 1.27. When $0 \leq \xi < 0.036$, the optimal credit limit increases with the enforcement rate. But when $0.036 \leq \xi < 0.04$, the optimal credit limit decreases with the enforcement rate. When $\xi \geq 0.04$, the social welfare in a credit equilibrium is lower than the welfare in a pure monetary equilibrium. When $\xi = 0$, the optimal enforcement rate is 68.33%, and it increases to 68.54% and then decreases to 68.52%, as ξ increases to 0.036 and 0.04.

4.4 Monetary policy

We show here that our main results hold when money supply grows at a constant rate $\gamma \ge 1$ and $M_{t+1} = \gamma M_t$. Note that the underlying friction that the government has no coercion power to tax agents implies that any deflationary monetary policy is not feasible. The monetary policy sets the lower bound of welfare that could be achieved in the economy, but it does not influence either the implementability or the optimal policy when a non-monetary equilibrium is optimal, which is when the cost parameter ξ is sufficiently low. The next proposition gives a sufficient condition such that the main results hold, where *i* denotes the nominal interest rate with $i = \gamma(1+r) - 1 \ge r$, and $\overline{z}(i)$ and $\Psi^m(i)$ denote the equilibrium real balances and value of using money given monetary policy *i* respectively:

Proposition 4.6 When $\xi \leq \min\{\frac{\sigma}{\sigma+r}\Psi^m(i), i\bar{z}(i)\}$, for any $\epsilon \in [0, 1]$, $D \in [\tilde{D}(\epsilon; \xi, i), \bar{D}(\epsilon; \xi)]$ is implementable and is not empty, with $\frac{d\bar{D}(\epsilon;\xi)}{d\epsilon} > 0$. The optimal welfare in the pure credit equilibrium and optimal credit limit are $\mathcal{W}^c(\xi)$ and $D^*(\xi)$, are independent with *i*.

Note that monetary policy i does not affect the upper bound of the implementable credit limit $\overline{D}(\epsilon; \xi)$ in a pure credit equilibrium, therefore does not affect the results in the welfare analysis when a pure credit equilibrium is optimal. The monetary policy i, however, does influence the implementable credit limit in a coexistence equilibrium, but this equilibrium is never socially optimal to choose. Moreover, the monetary policy i also influences the threshold of credit limit below which the buyer prefers money even though credit is provided, $\widetilde{D}(\epsilon; \xi, i)$, but in the welfare analysis, we only consider the highest implementable credit limit that would maximize the social welfare.

5 Concluding Remarks

We developed a model of endogenous use of unsecured credit and money, but under costly enforcement. We obtained three main results. First, the use of credit can be sustained in an equilibrium that is both incentive compatible and fiscally feasible only if the marginal cost of enforcement is not too high. Second, even when it is sustainable, a high enforcement rate may not be necessarily the optimal one, consistent with the empirical finding of the reverse U-shaped relationship between financial development and economic development; moreover, for a range of marginal cost of enforcement, it is optimal to use money alone while it is incentive compatible and fiscally feasible to sustain the use of credit. Third, there can be a non-monotonic relationship between the optimal credit limit and enforcement levels. These results suggest that by looking at the debt-GDP ratio alone or at indexes for institution qualities, which are typically proxies for efficiency of enforcement, as the main guidance for policy recommendations on different use of means-of-payments, can be misleading.

Although the optimal arrangement in our model is either a pure-currency economy or a pure-credit economy, one could potentially include other features into our baseline model to obtain coexistence, such as a two-stage DM structure as in Araujo and Hu (2018) with different costs of using the monitoring technology across meetings between the two DM stages. Our results can be useful as they point out the basic trade-offs, both for individuals and for the society as a whole.

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Appendix A: Proofs of Lemmas and Propositions

Proof of Lemma 3.1

We prove Lemma 3.1 by three steps:

- (1) There exists $\underline{z} \ge 0$ such that, if $\underline{z} > 0$, then $V^m(z) > V^c(z)$ if and only if $z > \underline{z}$; if $\underline{z} = 0$, then $V^m(z) \ge V^c(z)$ for all z with strict inequality for all z > 0.
- (2) $\Psi^{c}(D) \ge \Psi^{m}$ if and only if $\chi \le \overline{\chi}$ and $D \ge \widetilde{D}$, where \widetilde{D} is the unique solution to $\Psi^{c}(D) = \Psi^{m}$.
- (3) The buyer chooses to access the financial system if and only if $\Psi^c(D) \ge \Psi^m$.

(1) Let

$$f(z; D, \chi) = V^{c}(z) - V^{m}(z) = \sigma[u(y^{c}(z, D) - y^{c}(z, D)] - \sigma\chi - \sigma[u(y^{m}(z)) - y^{m}(z)]$$

denote the difference between the buyer's DM continuation value by accessing the financial system or not. It then follows that $f(z; D, \chi)$ strictly decreases with z for $z \leq y^*$, with $f(0; D, \chi) = \sigma[u(D) - D] - \sigma\chi$ and $f(y^*; D, \chi) = -\sigma\chi < 0$. We consider two cases. First, suppose that $\chi \leq u(D) - D$. Then, there exists a unique $\underline{z}(D, \chi) \in [0, y^*]$ such that $f(\underline{z}; D, \chi) = 0$. Second, suppose that $\chi > u(D) - D$. Then, let $\underline{z} = 0$. It follows that $V^m(z) > V^c(z)$ for all $z > \underline{z}(D, \chi)$, and, for the first case, $V^m(z) < V^c(z)$ for all $z < \underline{z}$. Moreover, $\underline{z}(D, \chi)$ increases in D and decreases in χ as f increases in D and decreases in χ .

(2) From equation (7), $\Psi^c(D) = -r(\bar{z} - D) + \sigma[u(\bar{z}) - \bar{z}]$ if $D < \bar{z}$, and $\Psi^c(D) = \sigma[u(D) - D]$ if $\bar{z} \leq D \leq y^*$. Hence, $\Psi^c(D)$ strictly increases with D for all $0 \leq D \leq y^*$. From equation (6), $\Psi^m = -r\bar{z} + \sigma[u(\bar{z}) - \bar{z}]$, which does not depend on D. Therefore, $\Psi^c(D) - \Psi^m$ strictly increases with D, with the maximum value $\Psi^c(y^*) - \Psi^m = \sigma[u(y^*) - y^*] - \sigma\chi - \Psi^m$ and the minimum value $\Psi^c(0) - \Psi^m = -\sigma\chi < 0$. By the Intermediate Value Theorem, when $\Psi^{c}(y^{*}) - \Psi^{m} \ge 0$, or equivalently

$$\chi \leqslant \frac{\sigma[u(y^*) - y^*] - \Psi^m}{\sigma} \equiv \bar{\chi},$$

there exists a unique $\tilde{D} \in [0, y^*]$ that solves $\Psi^c(D) = \Psi^m$, and $\Psi^c(D) \ge \Psi^m$ if and only if $D \ge \tilde{D}$. Otherwise, $\Psi^c(D) < \Psi^m$ for any $0 \le D \le y^*$.

(3) Combine equation (2) and (5), we can rewrite the buyer's CM decision as

$$\max_{z} \{ -rz + \sigma \max\{V^{m}(z), V^{c}(z)\} \}.$$
(19)

We show that problem (19) has same solution as

$$\max\{\Psi^{m}, \Psi^{c}(D)\} = \max\{\max_{z}\{-rz + \sigma V^{m}(z)\}, \max_{z}\{-rz + \sigma V^{c}(z)\}\}, \quad (20)$$

by considering three cases.

(3.a) $D < \tilde{D}$. In this case, the solution to (20) is such that $\Psi^m > \Psi^c(D)$ and $z = \bar{z}$. Note that $\bar{z} > \underline{z}(D, \chi)$ because

$$f(\bar{z}; D, \chi) = -r\bar{z} + \sigma[u(y^{c}(\bar{z}, D)) - y^{c}(\bar{z}, D)] - \sigma\chi - \Psi^{m} < \Psi^{c}(D) - \Psi^{m} < 0.$$

The solution to (19) is also $z = \overline{z}$: if $z > \underline{z}(D, \chi)$, then $V^m(z) > V^c(z)$ and $\max\{-rz + \sigma V^m(z)\} = \Psi^m$, which has solution $\overline{z} > \underline{z}(D, \chi)$; if $z \leq \underline{z}(D, \chi)$, then $V^m(z) \leq V^c(z)$ and $\max\{-rz + \sigma V^c(z)\} \leq \Psi^c(D) < \Psi^m$. Therefore, \overline{z} is also the unique solution of (19).

(3.b) $D \in [\tilde{D}, \max\{\bar{z}, \tilde{D}\})$. Then, the solution to (20) is such that $\Psi^m \leq \Psi^c(D)$ and $z = \bar{z} - D$. Note that $\bar{z} - D \leq \underline{z}(D, \chi)$ because

$$f(\bar{z} - D; D, \chi) = \Psi^{c}(D) - [-r(\bar{z} - D) + \sigma(u(\bar{z} - D) - (\bar{z} - D))] \ge \Psi^{c}(D) - \Psi^{m} > 0.$$

The solution to (19) is also $z = \overline{z} - D$: if $z \leq \underline{z}(D, \chi)$, then $V^c(z) \geq V^m(z)$, and $\max\{-rz + \sigma V^c(z)\} \leq \Psi^c(D)$ which has solution $z = \overline{z} - D \leq \underline{z}(D, \chi)$; if $z > \underline{z}(D, \chi)$, then $V^c(z) < V^m(z)$, and $\max\{-rz + \sigma V^m(z)\} = \Psi^m \leq \Psi^c(D)$. So the unique solution to (19) is also $\bar{z} - D$.

(3.c) $D \ge \max\{\bar{z}, \tilde{D}\}$. The solution to (20) is such that $\Psi^m \le \Psi^c(D)$ and z = 0. The solution to (19) is also z = 0: if $z \le \underline{z}(D, \chi)$, then $V^m(z) \le V^c(z)$ and $\max\{-rz + \sigma V^c(z)\} \le \Psi^c(D)$ which has solution $z = 0 \le \underline{z}(D, \chi)$; if $z > \underline{z}(D, \chi)$, then $V^m(z) \ge V^c(z)$ and $\max\{-rz + \sigma V^m(z)\} = \Psi^m \le \Psi^c(D)$. So the unique solution to equation (19) is the unique solution to equation (20).

Proof of Lemma 3.2

When z = 0, the consistency between the solution of $\max\{V^m(z), V^c(z)\}$ and $\max\{\Psi^m, \Psi^c(D)\}$ when z = 0 follows the same proof as in the proof of Lemma 3.1. As $\Psi^c(D) - \Psi^m = \sigma[u(D) - D - \chi]$ is a concave and increasing function on $D \in [0, y^*]$ with the minimum value $-\sigma\chi < 0$ and the maximum value $\sigma[u(y^*) - y^* - \chi]$, by the Intermediate Value Theorem, when $\chi \leq u(y^*) - y^* \equiv \hat{\chi}$, there exists a unique $\hat{D}(\chi) \in [0, y^*]$ solves $\Psi^c(\hat{D}) - \Psi^m = 0$, with $\hat{D}'(\chi) = 1/(u'(\hat{D}) - 1) > 0$, $\hat{D}''(\chi) = -u''(\hat{D})/(u'(\hat{D}) - 1)^2 > 0$.

Proof of Lemma 3.3

1. When $D \in [\tilde{D}, \bar{z})$, the DM output in the coexistence equilibrium is $y = \bar{z}$, and in the pure credit equilibrium is y = D. As $D < \bar{z}$, $\sigma[u(\bar{z}) - \bar{z} - \chi] > \sigma[u(D) - D - \chi]$.

2. When $D \in [\hat{D}, \tilde{D})$, the DM output in the pure monetary equilibrium is $y = \bar{z}$ and in the pure credit equilibrium is y = D. As $D < \tilde{D}$, from Lemma 3.1, $\sigma[u(D) - D - \chi] = \Psi^c(D) < \Psi^m = -r\bar{z} + \sigma[u(\bar{z}) - \bar{z}] < \sigma[u(\bar{z}) - \bar{z}].$

Proof of Lemma 3.4

Combine equation (8) and (9), we get equation (10) in Lemma 3.4. Rewrite equation (10) by a function $g(D; \epsilon, \xi)$ as

$$g(D;\epsilon,\xi) \equiv \Psi^{c}(D) - \frac{r}{\epsilon}D - \frac{r\xi}{\sigma} = \begin{cases} \Psi^{m} + rD - \xi\epsilon - \frac{r}{\epsilon}D - \frac{r\xi}{\sigma} & \text{if } 0 \leq D < \bar{z}; \\ \sigma(u(D) - D) - \xi\epsilon - \frac{r}{\epsilon}D - \frac{r\xi}{\sigma} & \text{if } \bar{z} \leq D < y^{*}; \\ \sigma[u(y^{*}) - y^{*}] - \xi\epsilon - \frac{r}{\epsilon}D - \frac{r\xi}{\sigma} & \text{if } D \geqslant y^{*}, \end{cases}$$

$$(21)$$

so the incentive constraint is equivalent as $g(D; \epsilon, \xi) \ge 0$. Notice that $g(D; \epsilon, \xi)$ is a decreasing function of D for any (ϵ, c) , with the maximum value $g(0; \epsilon, \xi) = \Psi^m - \xi \epsilon - \frac{r\xi}{\sigma}$ and the minimum value $g(\infty; \epsilon, \xi) = -\infty$. So there exists a unique $\bar{D} = \bar{D}(\epsilon; \xi)$ such that $g(\bar{D}; \epsilon, \xi) = 0$ if $\Psi^m - \xi \epsilon - \frac{r\xi}{\sigma} \ge 0$; otherwise let $\bar{D} < 0$. Then we have $g(D; \epsilon, \xi) \ge 0$ for all $D \le \bar{D}$.

Proof of Proposition 3.1

We prove in three steps:

- (1) For $\xi \leq \frac{\sigma}{\sigma+r}\Psi^m$, there exists $\bar{\epsilon}(\xi)$ such that $\tilde{D} \leq \bar{D}$ if and only if $\epsilon \in [0, \bar{\epsilon}(\xi)]$.
- (2) For any $\xi \leq \frac{\sigma}{\sigma+r}\Psi^m$ and any $D \leq \overline{D} \equiv \overline{D}(\overline{\epsilon}(\xi);\xi)$, there is a well-defined and continuously differentiable function $\epsilon(D;\xi)$, as the inverse of $\overline{D}(\epsilon;\xi)$, such that D is implementable if and only if $\epsilon(D;\xi) \leq \epsilon \leq \overline{\epsilon}(\xi)$.
- (3) Properties of $\overline{D}(\epsilon; \xi)$ in Proposition 3.1(2.a) and (2.b).

(1) From equation (21), $\tilde{D} \leq \bar{D}$ if and only if $g(\tilde{D}; \epsilon, \xi) \geq g(\bar{D}; \epsilon, \xi) = 0$. So we only need to show that there exists $\bar{\epsilon}(\xi)$ such that $g(\tilde{D}; \epsilon, \xi) \geq 0$ if and only if $\epsilon \in [0, \bar{\epsilon}(\xi)]$ for $\xi \leq \frac{\sigma}{\sigma+r} \Psi^m$. For $\xi > \frac{\sigma}{\sigma+r} \Psi^m$, we show that $g(\tilde{D}; \epsilon, \xi) < 0$ for all $\epsilon \in [0, 1]$.

First note that \tilde{D} increases in both ξ and ϵ . To see this, let $\Psi^m = \Psi^c(\tilde{D})$ as in equation (6) and (7): if $\xi \epsilon \leqslant r\bar{z}$, $\tilde{D} = \frac{\xi \epsilon}{r}$ and \tilde{D} increases with ϵ and ξ . If $\xi \epsilon > r\bar{z}$, \tilde{D} is determined by $\sigma[u(\tilde{D}) - \tilde{D}] = \Psi^m + \xi \epsilon$, and \tilde{D} increases with ϵ and ξ because $\sigma[u(D) - D]$ increases with D. Moreover, when $\xi \epsilon > r\bar{z}$, $\frac{\tilde{D}(\epsilon;\xi)}{\epsilon}$ is also increasing in ϵ because

$$\frac{\partial \frac{D(\epsilon;\xi)}{\epsilon}}{\partial \epsilon} = \frac{\sigma[u(\tilde{D}) - \tilde{D}] - \Psi^m - \sigma \tilde{D}(u'(\tilde{D}) - 1)}{\epsilon^2 \sigma[u'(\tilde{D}) - 1]} \ge \frac{\sigma[u(\bar{z}) - \bar{z}] - \Psi^m - r\bar{z}}{\epsilon^2 \sigma[u'(\tilde{D}) - 1]} = 0, \quad (22)$$

where the inequality uses the fact that $\tilde{D} \ge \bar{z}$ and $\sigma[u(D) - D] - \Psi^m - \sigma D(u'(D) - 1)$ is an increasing function of D.

Plug \tilde{D} into $g(D; \epsilon, \xi)$,

$$g(\tilde{D};\epsilon,\xi) = \begin{cases} \Psi^m - \frac{\sigma+r}{\sigma}\xi & \text{if } \xi\epsilon \leqslant r\bar{z}; \\ \Psi^m - \frac{r}{\epsilon}\tilde{D}(\epsilon;\xi) - \frac{r\xi}{\sigma} & \text{if } \xi\epsilon > r\bar{z}, \end{cases}$$
(23)

where in both cases $g(\tilde{D}; \epsilon, \xi)$ decreases in ξ and decreases in ϵ . Moreover, if $\xi \epsilon > r\bar{z}$, $g(\tilde{D}; \epsilon, \xi) < g(\tilde{D}; \frac{r\bar{z}}{\xi}, \xi) = \Psi^m - \frac{\sigma + r}{\sigma} \xi$. So for any $\epsilon \in [0, 1]$, $g(\tilde{D}; \epsilon, \xi) \leq \Psi^m - \frac{\sigma + r}{\sigma} \xi$. When $\xi > \frac{\sigma}{\sigma + r} \Psi^m$, it is straightforward to verify that $g(\tilde{D}; \epsilon, \xi) \leq \Psi^m - \frac{\sigma + r}{\sigma} \xi < 0$, so there is no implementable ϵ and $\bar{\epsilon}(\xi)$ is empty. Now we discuss $\bar{\epsilon}(\xi)$ when $\xi \leq \frac{\sigma}{\sigma + r} \Psi^m$ in

two cases:

(1.a) $r\bar{z} \ge \frac{\sigma}{\sigma+r}\Psi^m$. Given that $\xi \le \frac{\sigma}{\sigma+r}\Psi^m$, $\xi\epsilon \le \frac{\sigma}{\sigma+r}\Psi^m \le r\bar{z}$, and hence $g(\tilde{D};\epsilon,\xi) = \Psi^m - \frac{\sigma+r}{\sigma}\xi \ge 0$ for any $\epsilon \in [0,1]$. Thus, $\bar{\epsilon}(\xi) = 1$.

(1.b) $r\bar{z} < \frac{\sigma}{\sigma+r}\Psi^m$. We consider two subcases. If $\xi < r\bar{z}$, $\xi\epsilon < r\bar{z}$ for all $\epsilon \in [0, 1]$, and hence $g(\tilde{D}; \epsilon, \xi) = \Psi^m - \frac{\sigma+r}{\sigma}\xi > \Psi^m - \frac{\sigma+r}{\sigma}r\bar{z} > \Psi^m - \frac{\sigma+r}{\sigma}\frac{\sigma}{\sigma+r}\Psi^m = 0$. Thus, $\bar{\epsilon}(\xi) = 1$. If $r\bar{z} \leqslant \xi \leqslant \frac{\sigma}{\sigma+r}\Psi^m$, notice that when $\epsilon = 1$, $g(\tilde{D}; 1, \xi)$ decreases in ξ , with the maximum value $g(\tilde{D}; 1, r\bar{z}) = \Psi^m - r\bar{z} - \frac{r}{\sigma}r\bar{z} \ge 0$ and the minimum value $g(\tilde{D}; 1, \frac{\sigma}{\sigma+r}\Psi^m) = \Psi^m - r\tilde{D} - \frac{r}{\sigma}\Psi^m = -r\tilde{D} + \sigma[u(\tilde{D}) - \tilde{D}] - \Psi^m \leqslant 0$. So there exists a unique threshold $\dot{\xi} \in [r\bar{z}, \frac{\sigma}{\sigma+r}\Psi^m]$, determined by $g(\tilde{D}; 1, \dot{\xi}) = 0$, such that $g(\tilde{D}; 1, \xi) \ge$ 0 if $\xi \leqslant \dot{\xi}$, and $g(\tilde{D}; 1, \xi) < 0$ if $\xi > \dot{\xi}$. Then when $r\bar{z} \leqslant \xi \leqslant \dot{\xi}$, if $0 \leqslant \epsilon < \frac{r\bar{z}}{\xi}$, $g(\tilde{D}; \epsilon, \xi) = \Psi^m - \frac{\sigma+r}{\sigma}\xi \ge 0$, and if $\frac{r\bar{z}}{\xi} \leqslant \epsilon \leqslant 1$, $g(\tilde{D}; \epsilon, \xi) \ge g(\tilde{D}; 1, \xi) \ge 0$. So in the range $r\bar{z} \leqslant \xi \leqslant \dot{\xi}, \bar{\epsilon}(\xi) = 1$, and all $\epsilon \in [0, 1]$ is implementable. When $\dot{\xi} < \xi \leqslant \frac{\sigma}{\sigma+r}\Psi^m$, if $0 \leqslant \epsilon < \frac{r\bar{z}}{\xi}, g(\tilde{D}; \epsilon, \xi) = \Psi^m - \frac{\sigma+r}{\sigma}\xi \ge 0$. If $\frac{r\bar{z}}{\xi} \leqslant \epsilon \leqslant 1$, $g(\tilde{D}; \epsilon, \xi)$ decreases in ϵ , with the maximum value $g(\tilde{D}; \frac{r\bar{z}}{\xi}, \xi) = \Psi^m - \frac{\sigma+r}{\sigma}\xi \ge 0$, and the minimum value $g(\tilde{D}; 1, \xi) < 0$. So there exists a unique $\bar{\epsilon}(\xi) \in [\frac{r\bar{z}}{\xi}, 1)$ which solves $g(\tilde{D}; \bar{\epsilon}, \xi) = 0$, and $g(\tilde{D}; \epsilon, \xi) \ge 0$ for all $\epsilon \in [0, \bar{\epsilon}(\xi)]$. So in the range $\dot{\xi} < \xi \leqslant \frac{\sigma}{\sigma+r}\Psi^m$, only $\epsilon \in [0, \bar{\epsilon}(\xi)]$ is implementable.

(2) First we show that $r\tilde{D}(\epsilon;\xi) \ge \xi\epsilon^2$ for any $\epsilon \in [0,\bar{\epsilon}]$. We consider two cases. If $\xi\epsilon < r\bar{z}$, $\tilde{D} = \frac{\xi\epsilon}{r}$ and $r\tilde{D} - \xi\epsilon^2 = \xi\epsilon(1-\epsilon) \ge 0$. If $\xi\epsilon \ge r\bar{z}$, \tilde{D} is determined by $\sigma[u(\tilde{D}) - \tilde{D}] - \xi\epsilon = \Psi^m$. Thus, $r\tilde{D} - \xi\epsilon^2 > r\tilde{D} - \xi\epsilon = r\tilde{D} - \sigma[u(\tilde{D}) - \tilde{D}] + \Psi^m \ge r\bar{z} - \sigma[u(\bar{z}) - \bar{z}] + \Psi^m = 0$. It then follows that for any $\epsilon \in [0,\bar{\epsilon}]$, $\frac{\partial g(D;\epsilon,\xi)}{\partial \epsilon} = -\xi + \frac{rD}{\epsilon^2} \ge -\xi + \frac{r\tilde{D}}{\epsilon^2} \ge 0$.

Given $\epsilon = 1$, first-best is implementable if $g(y^*; 1, \xi) \ge 0$, which has solution $\xi < \xi^* \equiv \frac{\sigma}{\sigma + r} [\sigma(u(y^*) - y^*) - ry^*]$. Note that $\xi^* \ge 0$ if and only if $r \le \frac{\sigma(u(y^*) - y^*)}{y^*}$; otherwise if $r > \frac{\sigma(u(y^*) - y^*)}{y^*}$, y^* is never implementable even for $\xi = 0$. Given any $r \le \frac{\sigma(u(y^*) - y^*)}{y^*}$ and $\xi < \xi^*$, y^* is implementable if $g(y^*; \epsilon, \xi) \ge 0$ which has solution $\epsilon \ge \epsilon^*(\xi)$. Moreover, $\epsilon^*(\xi)$ increases in ξ . We define

$$\bar{\bar{\epsilon}}(\xi) = \begin{cases} \epsilon^*(\xi) & \text{if } \xi \leqslant \xi^*; \\ \bar{\epsilon}(\xi) & \text{if } \xi^* < \xi \leqslant \frac{\sigma}{\sigma+r} \Psi^m. \end{cases}$$
(24)

Note that $\overline{\overline{D}}(\xi) = \overline{D}(\overline{\overline{\epsilon}}(\xi);\xi) = \overline{D}(\overline{\epsilon}(\xi);\xi)$ as the highest implementable credit limit for given ξ . Thus,

$$\bar{\bar{D}}(\xi) = \begin{cases} y^* & \text{if } \xi \leqslant \xi^*; \\ \bar{D}(\bar{\epsilon}(\xi);\xi) < y^* & \text{if } \xi^* < \xi \leqslant \frac{\sigma}{\sigma+r} \Psi^m. \end{cases}$$
(25)

As when $\epsilon = \overline{\epsilon}$, \overline{D} is implementable, so $g(\overline{D}; \overline{\epsilon}, \xi) \ge 0$. For any $D \le \overline{D}(\xi)$, $g(D; \epsilon, \xi)$ strictly increases in ϵ for $\epsilon \le \overline{\epsilon}$, with the minimum value $g(D; 0, \xi) = -\infty$ and the maximum value $g(D; \overline{\epsilon}, \xi) \ge g(\overline{D}; \overline{\epsilon}, \xi) \ge 0$. By the Intermediate Value Theorem, there exists a unique $\epsilon(D; \xi) \in [0, \overline{\epsilon}]$ such that $g(D; \epsilon(D; \xi), \xi) = 0$. When $\epsilon \ge \epsilon(D; \xi)$, $g(D; \epsilon, \xi) \ge 0$. As for $D \in [0, \overline{D}]$, $g(D; \epsilon, \xi)$ is a continuously differentiable function on ϵ with positive partial derivative w.r.t ϵ , hence $\epsilon(D; \xi)$ is continuously differentiable.

(3) From equation (21), we define the enforcement rate when $\overline{D} = \overline{z}$ as $\underline{\epsilon}(\xi)$, which is determined by $g(\overline{z}; \underline{\epsilon}(\xi), \xi) = \Psi^m + r\overline{z} - \xi \underline{\epsilon} - \frac{r\overline{z}}{\underline{\epsilon}} - \frac{r\xi}{\sigma} = 0$ with $\underline{\epsilon}'(\xi) = \frac{\underline{\epsilon}^2(r/\sigma + \underline{\epsilon})}{r\overline{z} - \xi \underline{\epsilon}^2} > 0$, as $r\overline{z} \ge r\widetilde{D} \ge \xi \epsilon$. To compute the derivatives, we use the Implicit Function Theorem and

solve it by taking first-order derivative of $g(\bar{D}(\epsilon;\xi);\epsilon,\xi) = 0$ with respect to ϵ , which implies

$$\frac{d\bar{D}}{d\epsilon}\Big|_{0\leqslant\epsilon<\underline{\epsilon}} = \frac{\frac{r}{\epsilon}\bar{D}-\xi\epsilon}{r(1-\epsilon)} > 0 \text{ and } \frac{d\bar{D}}{d\epsilon}\Big|_{\underline{\epsilon}\leqslant\epsilon<\overline{\epsilon}} = \frac{\frac{r}{\epsilon}\bar{D}-\xi\epsilon}{r-\epsilon\sigma[u'(\bar{D})-1]} > 0;$$
(26)

where the inequality comes from $\frac{r}{\epsilon}\overline{D} - \xi\epsilon > \frac{r}{\epsilon}\widetilde{D} - \xi\epsilon$ and $r\widetilde{D} \ge \xi\epsilon$. Similarly, we take first-order derivative of $g(\overline{D}(\epsilon;\xi);\epsilon,\xi) = 0$ with respect to ξ to obtain

$$\frac{d\bar{D}}{d\xi}\Big|_{0\leqslant\epsilon<\underline{\epsilon}} = \frac{-\frac{r}{\sigma}-\epsilon}{r(1-\epsilon)} < 0; \text{ and } \frac{d\bar{D}}{d\xi}\Big|_{\underline{\epsilon}\leqslant\epsilon<\overline{\epsilon}} = \frac{-\frac{r}{\sigma}-\epsilon}{r-\epsilon\sigma[u'(\bar{D})-1]} < 0.$$
(27)

Finally, when $\epsilon = 0$, it is obvious that $\overline{D}(0;\xi) = 0$. When $\epsilon = \overline{\epsilon}(\xi)$, given $\xi \leq \frac{\sigma}{\sigma+r}\Psi^m$, by some algebra we can show that $\underline{\epsilon}(\xi) \leq \underline{\epsilon}(\frac{\sigma}{\sigma+r}\Psi^m) = \overline{\epsilon}(\xi)$, so $\overline{D}(\overline{\epsilon}(\xi);\xi) \geq \overline{D}(\underline{\epsilon}(\xi);\xi) = \overline{z}$.

Proof of Proposition 4.1

(1): When $D < \bar{z}$, $y^c = \bar{z}$ and $\mathcal{W}^c = \sigma[u(\bar{z}) - \bar{z}] - \xi \epsilon(D; \xi)$ where $\epsilon(D; \xi)$ is a strictly increasing function in D, (see Proposition 3.1(2.a)). Thus, $\frac{d\mathcal{W}^c}{dD} = -\xi \frac{d\epsilon}{dD} < 0$.

(2): When $D \ge \bar{z}, y^c = D$. Rewrite equation (15) as

$$\frac{d\mathcal{W}^c}{dD} = \frac{r}{rD - \xi\epsilon^2} [\sigma D(u'(D) - 1) - \xi\epsilon(D;\xi)],\tag{28}$$

where $rD - \xi\epsilon^2 > 0$ (see the proof of Proposition 3.1(2)), and the sign of $\frac{dW^c}{dD}$ depends on $\sigma D(u'(D) - 1) - \xi\epsilon(D;\xi)$. Notice that since $u(D) = \frac{D^{\alpha}}{\alpha}$, $\sigma D(u'(D) - 1)$ strictly decreases in D and $\epsilon(D;\xi)$ strictly increases in D. So $\sigma D(u'(D) - 1) - \xi\epsilon(D;\xi)$ decreases in $D \in [\bar{z}, \bar{\bar{D}}(\xi)]$, with maximum value $\sigma \bar{z}(u'(\bar{z}) - 1) - \xi\epsilon(\bar{z};\xi) = r\bar{z} - \xi\epsilon(\bar{z};\xi) > 0$ (see the proof of Proposition 3.1(2)), and the minimum value $\sigma \bar{\bar{D}}(\xi)(u'(\bar{\bar{D}}(\xi)) - 1) - \xi\epsilon(\bar{\bar{D}}(\xi);\xi)$. Now we show that $\sigma \bar{\bar{D}}(\xi)(u'(\bar{\bar{D}}(\xi)) - 1) - \xi\epsilon(\bar{\bar{D}}(\xi);\xi) < 0$ in three subcases: (2.a) $\xi \leq \xi^*$. From equation (25), $\bar{\bar{D}}(\xi) = y^*$, so $\sigma \bar{\bar{D}}(u'(\bar{\bar{D}}) - 1) - \xi\epsilon(\bar{\bar{D}};\xi) = -\xi\epsilon(y^*;\xi) < 0$. (2.b) $\xi^* < \xi \leqslant \dot{\xi}$. From equations (24) and (25), $\bar{\bar{D}} < y^*$ and $\epsilon(\bar{\bar{D}}, \xi) = 1$. Together with the incentive constraint $\sigma[u(\bar{\bar{D}}) - \bar{\bar{D}}] - \xi = r\bar{\bar{D}} + \frac{r\xi}{\sigma}$, we rewrite $\sigma\bar{\bar{D}}(u'(\bar{\bar{D}}) - 1) - \xi\epsilon(\bar{\bar{D}};\xi) = \frac{\sigma}{\sigma+r}[(\sigma+r)\bar{\bar{D}}u'(\bar{\bar{D}}) - \sigma u(\bar{\bar{D}})]$, which decreases in $\bar{\bar{D}} \in [\bar{z}, y^*)$, so $\sigma\bar{\bar{D}}(u'(\bar{\bar{D}}) - 1) - \xi\epsilon(\bar{\bar{D}};\xi) \leqslant \frac{\sigma}{\sigma+r}[(\sigma+r)\bar{z}u'(\bar{z}) - \sigma u(\bar{z})] < 0$.

(2.c) $\dot{\xi} < \xi \leqslant \frac{\sigma}{\sigma+r} \Psi^m$. From the proof of Proposition 3.1(1.b), $\xi \epsilon(\bar{\bar{D}};\xi) > r\bar{z}$, so $\sigma \bar{\bar{D}}(u'(\bar{\bar{D}})-1) - \xi \epsilon(\bar{\bar{D}};\xi) < \sigma \bar{\bar{D}}(u'(\bar{\bar{D}})-1) - r\bar{z} < \sigma \bar{z}(u'(\bar{z})-1) - r\bar{z} = 0.$

Therefore, by the Intermediate Value Theorem, there exists $D \in [\bar{z}, \bar{\bar{D}}(\xi)]$, below which $\frac{dW^c}{dD} > 0$, and above which $\frac{dW^c}{dD} < 0$.

Proof of Proposition 4.2

By Proposition 4.1(1), $\mathcal{W}^{c}[D, \epsilon(D; \xi)]$ strictly decreases in D for $D < \bar{z}$. Thus, we consider only $D \in [\bar{z}, \bar{\bar{D}}(\xi)]$. Moreover, for $\xi > \frac{\sigma}{\sigma+r}\Psi^{m}$, there is no implementable D. So we only consider $\xi \leq \frac{\sigma}{\sigma+r}\Psi^{m}$. In this range, $\mathcal{W}^{c}[D, \epsilon(D; \xi)] = \sigma[u(D) - D] - \xi\epsilon(D; \xi)$. Proposition 3.1(2) implies that $\mathcal{W}^{c}[D, \epsilon(D; \xi)]$ is continuously differentiable on (D, ξ) for $\epsilon \in [\underline{\epsilon}(\xi), \overline{\epsilon}(\xi)]$. By the Theorem of Maximum, $\mathcal{W}^{c}(\xi)$ is continuous in ξ . Moreover, $\overline{\bar{D}}(\xi)$ is continuously differentiable for $\xi \in [0, \xi^{*})$ and $\xi \in (\xi^{*}, \frac{\sigma}{\sigma+r}\Psi^{m}]$. Now, by the Envelope Theorem,

$$\frac{\mathcal{W}^c(\xi)}{d\xi} = -\xi \frac{d\epsilon}{d\xi} - \epsilon + \lambda \frac{d\bar{D}}{d\xi} < 0,$$
(29)

where $\lambda > 0$, for $\xi < \xi^*$ and $\xi^* < \xi \leq \frac{\sigma}{\sigma+r}\Psi^m$. This, together with the continuity of \mathcal{W}^c , implies that $\mathcal{W}^c(\xi)$ strictly decreases in ξ for all $\xi \in [0, \frac{\sigma}{\sigma+r}\Psi^m]$.

Now, when $\xi = 0$, $D^*(0) = y^* > \bar{z}$ and hence $\mathcal{W}^c(0) > \mathcal{W}^m$. When $\xi = \frac{\sigma}{\sigma + r} \Psi^m$, the only implementable $D = \overline{\bar{D}}(\frac{\sigma}{\sigma + r} \Psi^m) = \bar{z}$, so at optimal $D^*(\frac{\sigma}{\sigma + r} \Psi^m) = \bar{z}$ and

$$\mathcal{W}^{c}(\frac{\sigma}{\sigma+r}\Psi^{m}) = \sigma[u(\bar{z}) - \bar{z}] - \frac{\sigma}{\sigma+r}\Psi^{m}\bar{\epsilon}(\bar{z};\frac{\sigma}{\sigma+r}\Psi^{m}) < \mathcal{W}^{m}.$$

Thus, there exists a unique $\hat{\xi} \in [0, \frac{\sigma}{\sigma+r}\Psi^m]$ such that $\mathcal{W}^c(\hat{\xi}) = \mathcal{W}^m$.

Proof of Proposition 4.3

From Proposition 4.2, when $\xi \leq \hat{\xi}$, $\mathcal{W}^c(\xi) = \sigma[u(D) - D] - \xi \epsilon(D; \xi)$ and is twice continuously differentiable. The ranges of the choice variable $D \in [\bar{z}, \bar{\bar{D}}(\xi)]$ are both weakly decreasing in ξ . Moreover, there exists $\tilde{\xi}$ such that $\frac{\partial^2 \mathcal{W}^c(\xi)}{\partial D \partial \xi} < 0$ for $\xi < \tilde{\xi}$, because $\frac{\partial^2 \mathcal{W}^c(\xi)}{\partial D \partial \xi}$ is continuous on ξ and when $\xi = 0$:

$$\frac{\partial^2 \mathcal{W}^c(\xi)}{\partial D \partial \xi}|_{\xi=0} = \left(-\frac{\partial \epsilon}{\partial D} - \xi \frac{\partial^2 \epsilon}{\partial D \partial \xi}\right)|_{\xi=0} = -\frac{\partial \epsilon}{\partial D}|_{\xi=0} = -\frac{r\epsilon - \epsilon^2 \sigma [u'(D) - 1]}{rD} < 0.$$
(30)

Then by supermodularity, the optimal credit limit $D^*(\xi)$ decreases in ξ when $\xi \leq \tilde{\xi}$. Given functional form $u(y) = \frac{y^{\alpha}}{\alpha}$ $(0 \leq \alpha \leq 1)$ and $\xi \leq \hat{\xi}$, we discuss the optimal credit limit $D^*(\xi)$ in two cases:

1. $r < \sigma \frac{1-\alpha}{\alpha}$. From Proposition 4.1(2), social welfare first increases in D and then decreases in D, so for any $\xi \in [0, \hat{\xi}]$, the optimal credit limit is uniquely determined by FOC, $\sigma D^*[u'(D^*) - 1] = \xi \epsilon(D^*, \xi)$, with $\frac{dD^*}{d\xi} = \frac{\epsilon + c \frac{d\epsilon}{d\xi}}{\sigma[u'(D^*) - 1 + D^*u''(D^*)] - \xi \frac{d\epsilon}{dD^*}} < 0.$

2. $r \ge \sigma \frac{1-\alpha}{\alpha}$. From Proposition 3.1, for any $\xi \le \hat{\xi}$, $\bar{\epsilon}(\xi) = 1$ because $r\bar{z} \ge \frac{\sigma}{\sigma+r} \Psi^m$. Then by the similar method as in the proof of Proposition 4.1(2.b), social welfare in a pure credit equilibrium now increases in D as $\frac{dW^c[D,\epsilon(D;\xi)]}{dD} > 0$ for all $D \in [\bar{z}, \bar{\bar{D}}(\xi)]$. So the optimal credit limit is $D^*(\xi) = \bar{\bar{D}}(\xi)$, which decreases with ξ as $\frac{d\bar{D}}{d\xi} = \frac{d\bar{D}(1;\xi)}{d\xi} < 0$ as in equation (27).

Proof of Proposition 4.4

Rewrite the incentive constraint as in equation (21): $g(D; \epsilon, \xi) = \Psi^c(D) - \frac{r}{\epsilon}D - \frac{r}{\sigma}\xi v(\epsilon)$, which is a decreasing function on D with minimum value negative. We prove that $\xi \leq \min\{\frac{1}{v'(1)}\frac{\sigma}{\sigma+r}\Psi^m, r\bar{z}\}$ is a sufficient condition for implementability by showing that $g(\tilde{D}; \epsilon, \xi) \ge 0$ for any $\epsilon \in [0, 1]$. Note that given $\xi \leq r\bar{z}$ and the assumption v(1) = 1, $\xi v(\epsilon) \leq r\bar{z}$ for any ϵ . So $\tilde{D} = \frac{\xi v(\epsilon)}{r}$ and $g(\tilde{D}; \epsilon, \xi) = \Psi^m - \frac{\sigma+r}{\sigma}\frac{\xi v(\epsilon)}{\epsilon} > \Psi^m(1 - \frac{v'(\epsilon)}{v'(1)}) \ge 0$.

The highest implementable credit limit
$$\bar{D}$$
 increases with ϵ as: when $\bar{D} < \bar{z}$, $\frac{d\bar{D}}{d\epsilon} = \frac{\Psi^c(\bar{D}) - \xi v'(\epsilon)(\epsilon + \frac{r}{\sigma})}{r(1-\epsilon)} \geqslant \frac{\Psi^c(\bar{D}) - \xi v'(\epsilon)(\epsilon + \frac{r}{\sigma})}{r(1-\epsilon)} = \frac{\Psi^m - \xi v'(\epsilon)(\epsilon + \frac{r}{\sigma})}{r(1-\epsilon)} \geqslant \frac{\Psi^m(1 - \frac{v'(\epsilon)}{v'(1)})}{r(1-\epsilon)} \geqslant 0$; and when $D \geqslant \bar{z}$, $\frac{d\bar{D}}{d\epsilon} = \frac{\Psi^c(\bar{D}) - \xi v'(\epsilon)(\epsilon + \frac{r}{\sigma})}{r - \sigma \epsilon(u'(\bar{D}) - 1)} \geqslant \frac{\Psi^m(1 - \frac{v'(\epsilon)}{v'(1)})}{r - \sigma \epsilon(u'(\bar{D}) - 1)} \geqslant 0$.

Proof of Proposition 4.5

The proof follows the same path as under linear cost. Optimal social welfare strictly decreases with ξ is proved by the Envelope Theorem as in equation (29), and with convex cost, $\frac{dW^c(\xi)}{d\xi} = -\epsilon v'(\epsilon) \frac{d\epsilon}{d\xi} + \lambda \frac{\bar{D}}{d\xi} - v(\epsilon) < 0$ with $\lambda > 0$. The optimal credit limit decreases with ξ when ξ is sufficiently small is proved by supermodularity as in equation (30), and with convext cost:

$$\frac{\partial^2 \mathcal{W}^c(\xi)}{\partial D \partial \xi}|_{\xi=0} = (-v'(\epsilon) \frac{\partial \epsilon}{\partial D} - \xi v'(\epsilon) \frac{\partial^2 \epsilon}{\partial D \partial \xi})|_{\xi=0} = -v'(\epsilon) \frac{\partial \epsilon}{\partial D}|_{\xi=0} < 0.$$

Proof of Proposition 4.6

Given $i \ge r$, the equilibrium real balances is $\bar{z}(i) \le \bar{z}$ which solves $\sigma[u'(\bar{z}(i))-1] = i$, and the continuation value of using money only is $\Psi^m(i) = -i\bar{z}(i) + \sigma[u(\bar{z}(i)) - \bar{z}(i)] \le \Psi^m$. The continuation value of using credit when $D < \bar{z}(i)$ is $\Psi^c(D; i) = \Psi^m(i) + iD - \xi\epsilon$; otherwise it is same as described in Lemma 3.4.

The incentive constraint is now: $g(D; \epsilon, \xi, i) = \Psi^c(D; i) - \frac{r}{\epsilon}D - \frac{r}{\sigma}\xi \ge 0$, which is either a strictly decreasing function on D, or a first increasing then decreasing function on D, depending on the value of ϵ and i. We will show that in both cases, given $\xi \le$ $\min\{\frac{\sigma}{\sigma+r}\Psi^m(i), i\bar{z}(i)\}, g(\tilde{D}(\epsilon; \xi, i); \epsilon, \xi, i) \ge 0$ for any $\epsilon \in [0, 1]$, therefore, there exists $\bar{D} > \tilde{D}$ such that $g(\bar{D}; \epsilon, \xi, i) = 0$. To prove it, note that if $\xi \le i\bar{z}(i), \tilde{D}(\epsilon; \xi, i) = \frac{\xi\epsilon}{i} \le$ $\bar{z}(i)$. So $g(\tilde{D}(\epsilon; \xi, i); \epsilon, \xi, i) = \Psi^m(i) - \frac{r}{i}\xi - \frac{r}{\sigma}\xi \ge \Psi^m(i) - \xi - \frac{r}{\sigma}\xi \ge 0$ as $\xi \le \frac{\sigma}{\sigma+r}\Psi^m(i)$.

To show that the welfare implication is as same as in without inflation, we need to prove that the highest implementable credit limit $\overline{D}(\xi) = \overline{D}(1;\xi)$ satisfies $\overline{D}(1;\xi) \ge \overline{z}(i)$ when $\xi \le \min\{\frac{\sigma}{\sigma+r}\Psi^m(i), i\overline{z}(i)\}$, which is shown by $g(\overline{z}(i); 1, \xi, i) = -r\overline{z}(i) + \sigma[u(\overline{z}(i)) - \overline{z}(i)] - \frac{\sigma+r}{\sigma}\xi \ge \Psi^m(i) - \frac{\sigma+r}{\sigma}\xi \ge 0.$